# Improved Bayesian Update Method for Components Failure Rates

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**Abstract:** This paper presents and explains the algorithm and the results of a novel, robust, fast and accurate Bayesian update methodology that has been developed at Leibstadt Nuclear Power Plant (KKL) for components failure rate calculation as typically required in the Probabilistic Safety Assessment (PSA). In PSA, component failure rates are typically modelled using probability density functions representing the uncertainty range of the failure rates. International generic failure rates (prior information) need to be coupled with plant-specific failure statistics (evidence) through a Bayesian update process to obtain a best estimate of plant-specific reliability parameters (posterior information). It was noticed, that commonly used numerical integration functions (e.g. MATLAB built-in function) often results in numerical instabilities when applied to the required integral for Bayesian updates. A more robust algorithm was developed to resolve these instabilities. This algorithm covers all probability density functions (pdfs) of interest in the nuclear industry. It uses a Simpsons 4th order integration scheme to carry out the numerical integration needed in the Bayesian update, while using an optimal discretization technique taking into account some useful characteristics of the still unknown posterior distribution. In some cases, the Bayesian update is performed analytically either by algebraic derivations or using the property of conjugation. The novel method is validated on practical industrial examples and benchmarked against the computational software Mathematica to prove its correctness and robustness.

**Keywords:** Probabilistic Safety Assessment, Bayesian Update, Numerical Integration, Smart Discretization.

## 1 INTRODUCTION

Bayesian updates are used extensively in machine learning, reliability analysis, and many other research and industrial areas in order to update the probability of a hypothesis when more evidence is available. In the nuclear industry, and specifically in PSA, the Bayesian update is used to amend a priori generic reliability data (e.g. component failure rate) with the plant-specific evidence [1]. This helps in achieving an optimal balance between international performance and plant-specific performance of components. Consequently, one gets a more accurate representation of plant-specific components' failure rates by optimally aggregating all the available information. The basic Bayes' theorem is given by

$$f(\lambda \mid E) = \frac{f(\lambda)L(E \mid \lambda)}{\displaystyle\int_0^\infty f(\lambda')L(E \mid \lambda')d\lambda'} \tag{1}$$

where $E$ is the evidence or the observed data, i.e. the recorded number of failures; $\lambda$ is the component failure rate (per time unit or per demand); $f(\lambda)$ is the prior's probability density function before observing $E$; $L(E \mid \lambda)$ is the likelihood function, i.e. it is a probability distribution of the number of failures; $f(\lambda \mid E)$ is the posterior probability density function,

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i.e. it is the state of knowledge of $\lambda$ after observing $E$.

Fig. 1 visually shows how a Bayesian update works: a prior distribution is aggregated with a likelihood distribution leading to a posterior distribution.

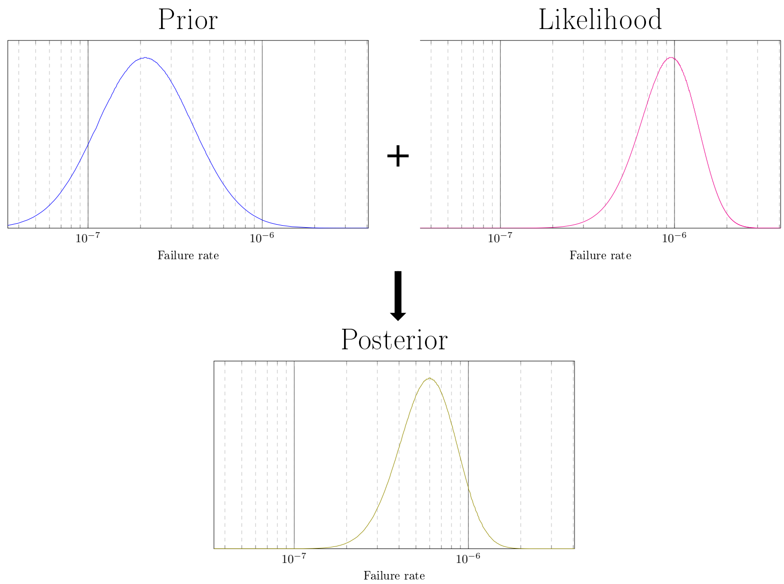



**Figure 1:** Illustrative Bayesian update example

It was observed that the evaluation of the denominator of Eq. 1 is challenging using traditional integration methods. Remarkable numerical instabilities were observed in a few cases, even with MATLAB, when evaluating the integral for commonly used distribution types. Examples of these instabilities are presented in Section 3.

At the Leibstadt nuclear power plant (KKL), the Bayesian update is performed following the requirements and the guidelines of the Swiss nuclear regulator ENSI. Guideline ENSI-A05, Chap. 4.2 [2] on the "Quality and Scope of PSA" requires that the plant-specific reliability parameters (in the following called *posterior* information) is obtained through a Bayesian update process. The guideline also requires the uncertainty distribution of the components failure probability, i.e. the mean value along with its associated percentiles. Eventually, the resulting uncertainty distribution can be mapped (*re-casting* procedure) into a definite continuous distribution type (e.g. Lognormal, Beta or Gamma distribution). As international generic data sources (in the following called *prior* information), KKL uses ZEDB, T-Book, NUREG/CR-6928, IAEA-TECDOC-478, EGG-SSRE-8875, WASH-1400. Plant specific evidence data includes plant component failure records and the exposure data (operating time or number of demands) as recorded by plant personnel. Plant specific data follows either a Poisson distribution, in case the exposure is considered per unit time, or a Binomial distribution, in case the exposure is considered per demand.

Gebraeel et al. [3] developed a Bayesian update approach for normally distributed priors and likelihoods. Zwirglmaier and Straub [4] derived a static discretization procedure using the First-Order Reliability Method. Kozlov and Koller [5] introduced the idea of dynamic discretization that searches for the most efficient discretization technique iteratively using the Junction Tree approach [6]. Besides, Neil et al. [7] added to the dynamic discretization technique a propagation algorithm on junction trees. Cobb and Shenoy [8] and Moral et al [9] resorted to the mixtures of truncated exponentials approach as an alternative to discretization. Murphy [10] introduced a variational approximation to perform the Bayesian inference. Furthermore, Shachter and Peot [11] and Carlin and Gelfand [12] used Monte Carlo sampling techniques to perform an approximate Bayesian inference.

The KKL PSA team previously developed a Bayesian update software based on an adaptive quadrature integration scheme using predefined logarithmic uniform discretization intervals, solely based on prior information. In some cases, it was observed that the likelihood distribution was shifted with respect to the prior, leading to suboptimal integration. Curtailments of posterior distribution were observed in specific cases, leading to underestimation or overestimation of the component failure rate. Furthermore, the algorithm did not distinguish between different prior distributions and their possible conjugability, performing always a numerical Bayesian update and therefore introducing a source of error due to numerical approximations. Finally, in the PSA documentation, as a result of those inaccuracies, the graphical representation of the Bayesian process was not always satisfactory.

In order to investigate those issues in more detail, a benchmark analysis was performed using the numerical integration function of MATLAB on Lognormal distributions. The aim was to compare the obtained values from the existing software with a reference software. It was noticed, that even using a reference code like MATLAB, the numerical integration function applied to the denominator of Eq. 1 often resulted in numerical instabilities, forcing the PSA team to develop a more robust and stable Bayesian update software. The numerical instabilities of MATLAB is presented in Section 3 along with the results of a stability check of the developed code.

In order to resolve some of the aforementioned issues and to improve the stability, a new algorithm was developed at KKL. The developed tool covers all probability density functions (pdfs) of interest in the nuclear industry. It covers prior data having a Lognormal, Normal, Gamma, Beta, Uniform, or Discrete[1] distribution, and likelihood data following a Binomial or a Poisson distribution. The Discrete distribution can be seen as a generalization of any other distribution type. The new tool performs analytic Bayesian updates when the distributions are conjugate[2], and resorts to a numerical approach that is stable, robust, fast, and efficient when an analytic solution does not exist.

The numerical integration uses a fourth order Simpsons [14] scheme with an adaptive discretization based on foreseen characteristics of the posterior distribution. At the heart of the novel approach, an innovative and optimal method to detect the characteristics of the posterior distribution before carrying out any integration step was developed and called the "Modal Method". The mode of the – still unknown – posterior is determined by setting the derivative of the distribution function $f(\lambda \mid E)$ to zero (Section 2). These improvements drastically simplify the required number of discretization points and allow an accurate capture of the posterior parameters (presented in Section 3). The tool performs a so called re-casting procedure in which the resulting discretely-defined posterior distribution is mapped to a known continuous parametric distribution. A criterion for the consistency of plant specific data and prior is introduced in the process of the evaluation of the posterior probability density function, following the recommendation of NUREG-CR/6823 [15]. The code is implemented in both Ruby script language and MATLAB.

[1] The Discrete distribution definition is adopted from RiskSpectrum: it is a piecewise constant (uniform) distribution.

[2] A prior distribution type is said to be *conjugate* for a likelihood function if the resulting distribution from a Bayesian update process is of the same type as the prior distribution. A rigorous mathematical definition of the concept of conjugability can be found in [13].

The software Mathematica is used to verify the results and prove their correctness. Moreover, Bayesian updates using Discrete prior distribution and the Modal Method are compared to prove their consistency. To the best of our knowledge, our new Bayesian update tool is faster and more robust compared to other techniques found in the literature, mainly due to its optimal discretization-scheme using the mode finding technique.

# 2 METHODOLOGY

The main aspects of the tool are described in this section. In what follows, the resolved cases, the general approach and the re-casting procedure are presented.

## 2.1. Resolved Cases

Table 1 lists all investigated combinations of prior and likelihood distribution along with their respective Bayesian update method.

**Table 1:** Considered Combinations

| Category | Case | Prior | Likelihood | Update Method |
|---|---|---|---|---|
| Non-Conjugate | 1 | Lognormal | Binomial | Numerical Integration (Modal Method) |
| | 2 | Lognormal | Poisson | Numerical Integration (Modal Method) |
| | 3 | Normal | Binomial | Numerical Integration (Modal Method) |
| | 4 | Normal | Poisson | Numerical Integration (Modal Method) |
| | 5 | Uniform | Binomial | Analytical Derivation |
| | 6 | Uniform | Poisson | Analytical Derivation |
| Conjugate | 7 | Gamma | Binomial | Transformation + Conjugation |
| | 8 | Gamma | Poisson | Conjugation |
| | 9 | Beta | Binomial | Conjugation |
| | 10 | Beta | Poisson | Transformation + Conjugation |
| Discrete | 11 | Discrete | Binomial | Analytical Derivation |
| | 12 | Discrete | Poisson | Analytical Derivation |

## 2.2. General Approach

The combinations listed in Table 1 can be classified into three categories: non-conjugate, conjugate, and Discrete. The non-conjugate distributions require solving Eq. (1) either numerically (cases 1-4) or analytically (cases 5-6). The conjugate distributions have known Bayesian update results that are obtained analytically. The Discrete is a special case that can be used to mimic any combination of other distributions. For the latter case, the developed tool performs a full analytical Bayesian update.

### 2.2.1. Non-Conjugate Distributions (cases 1-6)

In general, the prior and likelihood distributions do not belong to the same family and the posterior distribution has a mathematically different density function than the prior [13]. In this case, the prior and likelihood distributions are called "non-conjugate".

Numerical Bayesian Update using the Modal Method (cases 1-4)

A numerical based Bayesian Update is required in the cases of Normal and Lognormal prior distributions to be updated with either a Binomial or a Poisson likelihood distribution. The integration scheme used is the fourth order Simpsons [14], which carries out an integral on every three-point parabolic segments, and sums these partial integrals to obtain the total integral.

The developed discretization method (Modal Method) insures an optimal discretization of the high information region, i.e. the region around the peak (mode) of the posterior distribution. The method foresees the exact location where the mode of the posterior distribution resides (step 1) and, prior to any numerical integration step, defines smart discretization intervals (step 2).

Step 1: Determination of the mode of the posterior To determine the mode of the posterior distribution, the derivative of the (yet unknown) posterior probability density function is set to zero. Eq. (2) below is called "Modal Characteristic Equation".

$$\frac{d}{d\lambda}\left(f(\lambda \mid k)\right) = 0$$

$$\frac{d}{d\lambda}\left(\frac{f(\lambda)L(k \mid \lambda)}{\int_0^{\infty} f(\lambda^{'})L(k \mid \lambda^{'})d\lambda^{'}}\right) = 0$$

$$\frac{d}{d\lambda}\left(f(\lambda)L(k \mid \lambda)\right) = 0$$

$$L(k \mid \lambda)\frac{df(\lambda)}{d\lambda} + f(\lambda)\frac{dL(k \mid \lambda)}{d\lambda} = 0 \tag{2}$$

As an example, for a Lognormal prior distribution with a Binomial likelihood function, the "modal characteristic equation" results in:

$$(\lambda - 1)\ln(\lambda) + \lambda\left((1-n)\,\sigma^2 - \mu\right) + (k-1)\,\sigma^2 + \mu = 0 \tag{3}$$

A numerical solution of Eq. (2) allows us to know in advance (i.e. before the numerical integration is started) the precise location of the mode of the posterior.

Step 2: Pseudo posterior and smart discretization The aim of this step is to generate a preliminary posterior distribution (called "pseudo posterior") mimicking to the closest possible the posterior distribution to be obtained through the Bayesian update (Eq. (1)). The pseudo posterior distribution will then be used to generate a smart discretization pattern, leading to an accurate integration in the denominator of Eq. (1). Two parameters are used to characterize the pseudo posterior distribution: the mode of the posterior (as calculated by Eq. (2) in Step 1) and the variance. It is assumed that the posterior distribution has the shape of the prior, i.e. the same distribution type.

It was observed that the variance of the posterior distribution is always smaller than the variance of the prior[3]. This is an intuitive result, since the uncertainty distribution is reduced as more information is available. To guarantee the coverage of the entire domain needed in the Bayesian update calculations, the variance of the pseudo posterior distribution was conservatively defined equal to the variance of the prior. Note that the generated pseudo posterior – characterized by the mode and the variance – is only used for discretization.

[3] being $Var(\theta) = E[Var(\theta \mid y)] + Var(E[\theta \mid y])$ [16]

Finally, the smart discretization pattern is obtained with the help of the inverse cumulative density function (cdf), as visually described in Fig. 2. A set of uniform spaced points between 0 and 1 ($y$-axis) is evaluated with the inverse cdf to obtain a set of optimally spaced integration points on the $\lambda$ axis.

At this point, the integral in Eq. 1 can be numerically evaluated.

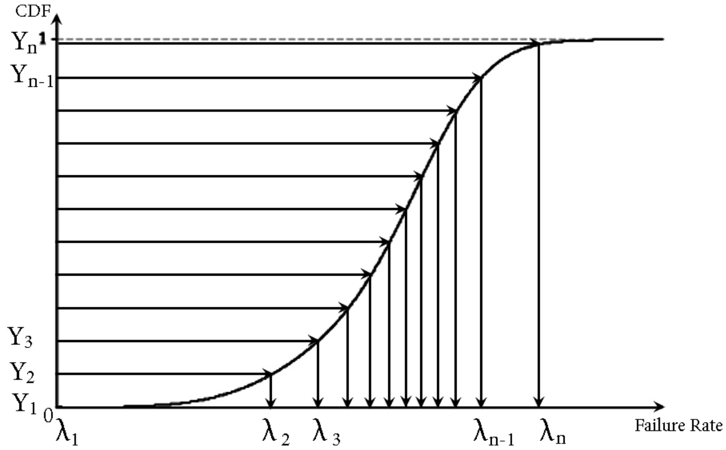



**Figure 2:** Discretization using the inverse cdf method

Table 2 lists the modal characteristic equations and the results of the Bayesian update for priors following a Lognormal or a Normal distribution.

Analytical Bayesian Update for Uniform Distributions (cases 5-6)

For the cases of a Uniform prior distribution to be updated with either a Binomial or a Poisson likelihood distribution, no discretization or numerical integration is needed, and a full analytical Bayesian update is done after analogizing a Poisson with a Gamma and a Binomial with a Beta distribution. The Uniform distribution is usually used if no information is known about the failure rate, in this case the prior distribution is called a non-informative prior distribution [17]. Table 3 summarizes the results of the Bayesian update for priors having a Uniform distribution.

#### 2.2.2. Conjugate Distributions (cases 7-10)

Four combinations of prior and likelihood distributions were identified as conjugate, hence the resulting posterior distribution and its parameters can be calculated analytically[4]. Table 4 summarizes the results of the Bayesian update for priors following a Gamma or a Beta distribution.

#### 2.2.3. Discrete Distributions (cases 11-12)

The Discrete case is a special case in which the prior distribution is defined as a piecewise constant function. This definition allows any distribution to be given in a discrete way in which

[4] The combination “Gamma with Binomial” and “Beta with Poisson” needed a transformation of variable followed by the application of the Poisson Limit Theorem [18] before applying the properties of conjugate distribution.

a number of piecewise constant segments can describe the shape and preserve the characteristics of that distribution. For a Discrete prior updated with either a Binomial or a Poisson likelihood distribution, a full analytical Bayesian update is performed and no discretization or numerical integration is needed. Table 5 summarizes the results of the Bayesian update for priors having a Discrete distribution.

### 2.3. Re-casting

The Bayesian update results in a posterior distribution having a non-defined distribution type (except for the special cases of conjugate distributions [13]). It is therefore often convenient to *re-cast* (fit) the posterior distribution into a known distribution type so that it becomes easy to handle in applications, rather than having to deal with a cumbersome discretized distribution (array of points). A re-casting procedure is therefore introduced and the posterior distribution is fit into a known distribution while preserving its main parameters (mean and median values). In some cases, the obtained $p_{95}$ percentiles are found to be slightly optimistic (i.e. $p_{95} < p_{95,\text{ exact}}$). To avoid this optimistic assignment, a refined re-casting procedure based on matching the mean ($\bar{x}$) and the 95th percentile ($p_{95,\text{ exact}}$) is preferred and implemented.

Specifically for the case of a Discrete prior distribution, the resulting posterior equation has the exact shape of the likelihood function in each interval, as shown in Table 5. The re-casting procedure is performed by optimally trying to preserve the mean and as many percentiles as possible, hence the re-casting problem is solved as a constrained optimization problem.

### 2.4. Prior to posterior consistency test

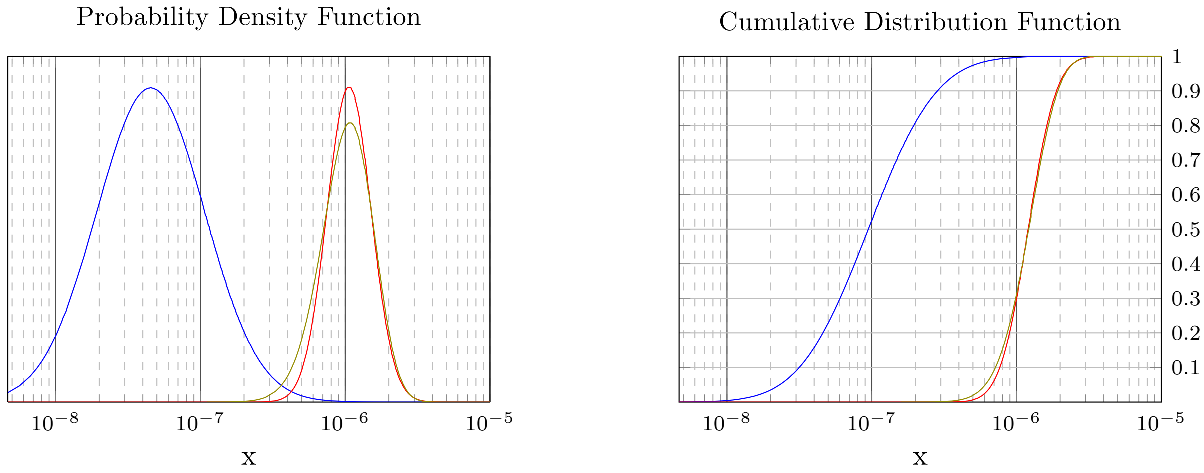
A criterion for the consistency of plant specific data and prior is introduced in the process of the evaluation of the posterior probability density function, following the recommendation of NUREG-CR/6823 [15], which states "If the observed data show x/t (number of failures over exposed time) very different from the prior mean, the analyst might wonder if the data and the prior are consistent, or if, instead, the prior distribution was *misinformed*". If the exact posterior mean exceeds the $p_{95}$ or is below the $p_{05}$ percentile of the prior distribution, a flag is raised and the user is informed about a potential inconsistency between prior and evidence information.

## 3 RESULTS

The developed algorithm was implemented in Ruby script language and showed remarkable efficiency and stability. Some representative examples were investigated and are discussed below.

Fig. 3 shows the plots and the table of properties of the prior, the exact posterior, and the re-cast posterior distributions obtained using the new Bayesian update tool applied on a prior distribution. The prior Lognormal distribution has a mean value of $1.37 \cdot 10^{-7}$ 1/h with an error factor of 4.1. The chosen parameters of the Poisson likelihood function are 9 failures over an exposure time of $4.34 \cdot 10^{6}$ h, delivering an "expected mean" of $2.1 \cdot 10^{-6}$ 1/h. Those rather extreme values were chosen such that the expected mean is heavily shifted towards higher failure rates, in order to test the code against the issues described in Section 1. The developed software tool, using the smart discretization algorithm, is able to calculate the mean value and the characteristic parameters of the posterior distribution correctly up to several significant digits. The results were cross-checked with software Mathematica.

A set of Bayesian updates were performed for the Lognormal distribution with a Poisson




The prior, posterior exact and posterior distributions are characterized by the following parameters:

| | Prior | Posterior exact | Posterior |
|---|---|---|---|
| **Mean** | 1.37E-07 | 1.28E-06 | 1.28E-06 |
| **Error Factor** | 4.10E+00 | | 1.78E+00 |
| **5 %-percentile** | 2.31E-08 | 6.05E-07 | 6.74E-07 |
| **50 %-percentile** | 9.47E-08 | 1.21E-06 | 1.20E-06 |
| **95 %-percentile** | 3.88E-07 | 2.15E-06 | 2.15E-06 |

**Figure 3: Bayesian update of a Lognormal prior distribution with a Poisson Likelihood function (9 failures over an exposure time of 4.34·$10^6$ h)**

likelihood function using the MATLAB built-in integral function. The prior mean varied between $10^{-6}$ and $10^{-4}$ 1/h) while the error factor ($EF = 10$), the number of failures ($k = 22$) and the exposure time ($T = 10^6$ h, $T = 10^7$ h) were held constant. The same analysis was performed with the developed tool. The results from both methods are plotted in the form of *Posterior mean* as function of the *Prior mean* and are presented in Fig. 4. Remarkable discontinuities appear when using the MATLAB integral function, leading to wrong Bayesian update results. On the other side, the developed method resulted in a very robust and stable Bayesian update, thanks to the smart adaptive discretization algorithm based on the "Modal Method".

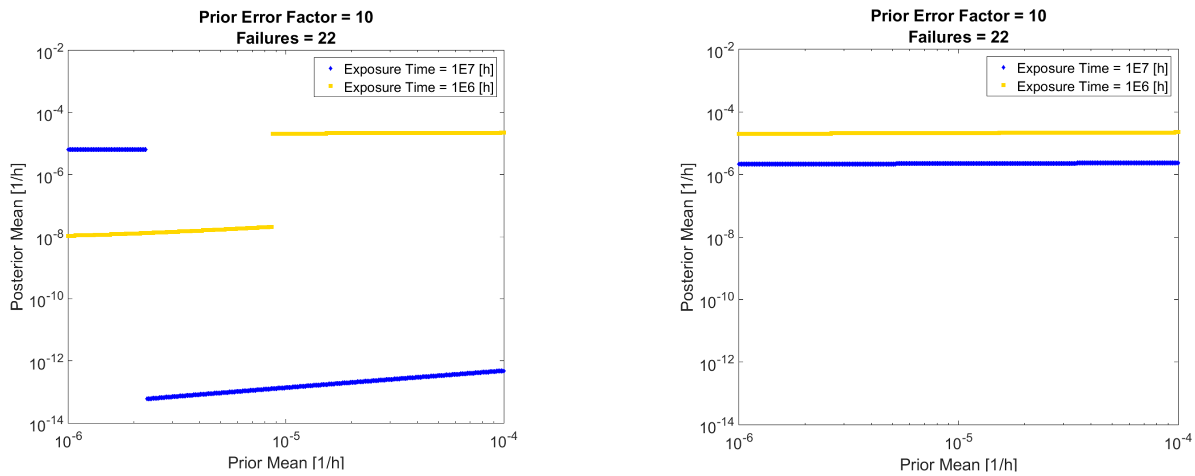
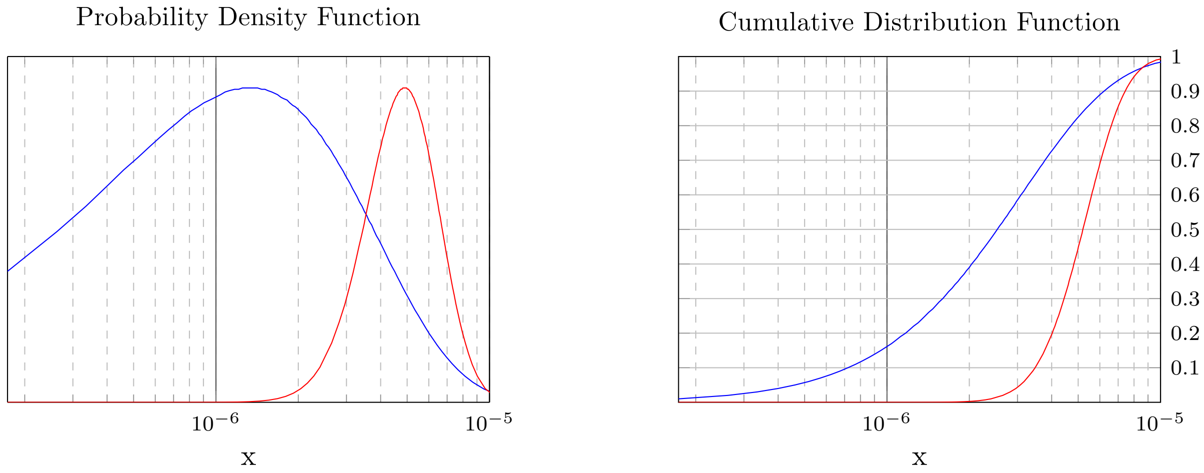



**Figure 4: MATLAB (left) and "Modal Method" (right) stability check for a Lognormal prior distribution with a Poission likelihood function.**

Fig. 5 shows the plots and the table of properties of the prior and the exact posterior




The prior and posterior distributions are characterized by the following parameters:

| | Prior | Posterior |
|---|---|---|
| **Mean** | 3.10E-06 | 5.35E-06 |
| **Alpha** | 1.75E+00 | 1.18E+01 |
| **Beta** | 5.65E+05 | 2.19E+06 |
| **5 %-percentile** | 4.60E-07 | 3.07E-06 |
| **50 %-percentile** | 2.53E-06 | 5.20E-06 |
| **95 %-percentile** | 7.67E-06 | 8.16E-06 |

**Figure 5: Bayesian update of a Gamma prior distribution with a Poisson Likelihood function (10 failures over an exposure time of $1.63{\cdot}10^6$ h)**

distributions for the case of a Gamma prior distribution. The prior Gamma distribution has a mean value of $3.1{\cdot}10^{-6}$ and a shape parameter $\alpha = 1.75$. The plant experience chosen for this example is 10 failures over an exposure time of $1.63{\cdot}10^6$ h.

A Bayesian update of a Lognormal & Binomial case was performed to compare the results of the Discrete method and the Modal Method. The corresponding plot in Fig. 6 shows the nice matching between the re-cast Discrete posterior and both the exact and re-cast posterior distributions obtained with the Modal Method.

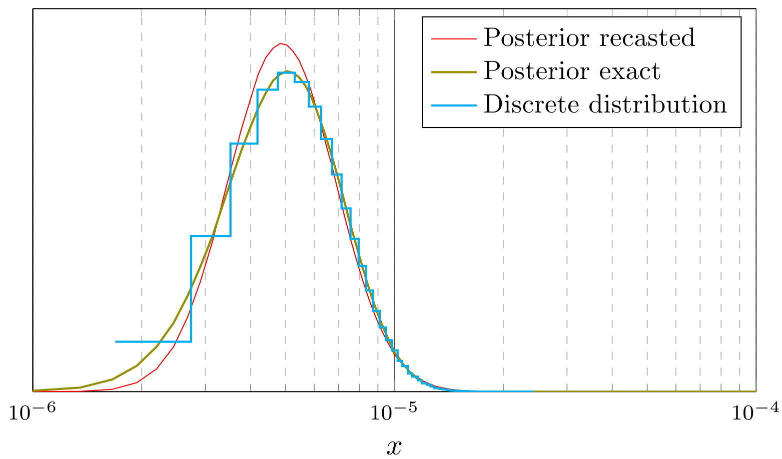



**Figure 6: Discrete and “Modal Method” Bayesian update comparison**

**Table 2: Summary table of the results for the Bayesian updates for numerical non-conjugable combinations**

| Prior Distribution ($\lambda$) | Likelihood Distribution ($k \mid \lambda$) | Modal Characteristic Equation | Posterior Distribution ($\lambda \mid k$) | Posterior mean $E[\lambda \mid k]$ | Posterior variance $Var[\lambda \mid k]$ | Posterior percentiles $p_n$ |
|---|---|---|---|---|---|---|
| $Lognormal(\mu,\sigma)$ | Binomial $(n,\lambda)$ | $(\lambda-1)\ln(\lambda)+\lambda((1-n)\sigma^2-\mu)+(k-1)\sigma^2+\mu=0$ | Numerically Calculating $\frac{f(\lambda)L(k \mid \lambda)}{\int_0^{\infty} f(\lambda)L(k \mid \lambda)d\lambda}$ | Numerically Integrating $\int_{-\infty}^{+\infty} \lambda f(\lambda \mid k)d\lambda$ | Numerically Integrating $\int_{-\infty}^{+\infty} (\lambda - E[\lambda])^2 f(\lambda \mid k)d\lambda$ | Numerically Solving $\int_{-\infty}^{p_n} f(\lambda \mid k)d\lambda = n\%$ |
| $Lognormal(\mu,\sigma)$ | Poisson $(\lambda T)$ | $\ln(\lambda)+(T\sigma^2)\lambda+(1-k)\sigma^2-\mu=0$ | | | | |
| $Normal(\mu,\sigma)$ | Binomial $(n,\lambda)$ | $\lambda^3-(\mu+1)\lambda^2+(\mu-n\sigma^2)\lambda+k\sigma^2=0$ | | | | |
| $Normal(\mu,\sigma)$ | Poisson $(\lambda T)$ | $\lambda^2+(T\sigma^2-\mu)\lambda-k\sigma^2=0$ | | | | |

**Table 3: Summary table of the results for the Bayesian updates for the analytical non-conjugable combinations**

| Prior Distribution ($\lambda$) | Likelihood Distribution ($k \mid \lambda$) | Exact posterior Distribution ($\lambda \mid k$) | Posterior mean $E[\lambda \mid k]$ | Posterior variance $Var[\lambda \mid k]$ | Posterior percentiles $p_n$ |
|---|---|---|---|---|---|
| $Uniform(a,b)$ | Binomial $(n,\lambda)$ | $\frac{\binom{n}{k}\cdot\lambda^k(1-\lambda)^{n-k}}{\frac{1}{n+1}\cdot(I_1(b)-I_1(a))}$ | $\frac{k+1}{n+2}\cdot\frac{(I_2(b)-I_2(a))}{(I_1(b)-I_1(a))}$ | $\frac{(k+2)(k+1)}{(n+3)(n+2)}\cdot\frac{(I_3(b)-I_3(a))}{(I_1(b)-I_1(a))}-(E[\lambda \mid k])^2$ | $I_1^{-1}(n\%\cdot I_1(b)+(1-n\%)\cdot I_1(a))$ |
| $Uniform(a,b)$ | Poisson $(\lambda T)$ | $\frac{\frac{(\lambda T)^k}{k!}\cdot e^{-\lambda T}}{\frac{1}{T}\cdot(\gamma_1(bT)-\gamma_1(aT))}$ | $\frac{(k+1)}{T}\cdot\frac{(\gamma_2(bT)-\gamma_2(aT))}{(\gamma_1(bT)-\gamma_1(aT))}$ | $\frac{(k+2)(k+1)}{T^2}\cdot\frac{(\gamma_3(bT)-\gamma_3(aT))}{(\gamma_1(bT)-\gamma_1(aT))}-(E[\lambda \mid k])^2$ | $\frac{\gamma_1^{-1}(n\%\cdot\gamma_1(bT)+(1-n\%)\cdot\gamma_1(aT))}{T}$ |

**Table 4: Summary table of the results for the Bayesian updates for the conjugable combinations**

| Prior Distribution ($\lambda$) | Likelihood Distribution ($k \mid \lambda$) | Exact posterior Distribution ($\lambda \mid k$) | Posterior mean $E[\lambda \mid k]$ | Posterior variance $Var[\lambda \mid k]$ | Posterior percentiles $p_n$ |
|---|---|---|---|---|---|
| $Gamma(\alpha,\beta)$ | Binomial $(n,\lambda)$ | $Gamma(\alpha+k,\beta+n)$ | $\frac{\alpha+k}{\beta+n}$ | $\frac{\alpha+k}{(\beta+n)^2}$ | $Gamma$ CDF Inverse |
| $Gamma(\alpha,\beta)$ | Poisson $(\lambda T)$ | $Gamma(\alpha+k,\beta+T)$ | $\frac{\alpha+k}{\beta+T}$ | $\frac{\alpha+k}{(\beta+T)^2}$ | $Gamma$ CDF Inverse |
| $Beta(\alpha,\beta)$ | $Binomial(n,\lambda)$ | $Beta(\alpha+k,\beta+n-k)$ | $\frac{\alpha+k}{\alpha+\beta+n}$ | $\frac{(\alpha+k)(\beta+n-k)}{(\alpha+\beta+n)^2(\alpha+\beta+n+1)}$ | $Beta$ CDF Inverse |
| $Beta(\alpha,\beta)$ | Poisson $(\lambda T)$ | $Beta(\alpha+k,\beta+T-k)$ | $\frac{\alpha+k}{\alpha+\beta+T}$ | $\frac{(\alpha+k)(\beta+T-k)}{(\alpha+\beta+T)^2(\alpha+\beta+T+1)}$ | $Beta$ CDF Inverse |

**Table 5: Summary table of some results for the Bayesian updates for the Discrete with a Poisson likelihood distribution**

| Prior Distribution ($\lambda$) | Likelihood Distribution($k \mid \lambda$) | Exact posterior Distribution ($\lambda \mid k$) | Posterior mean $E[\lambda \mid k]$ | Posterior variance $Var[\lambda \mid k]$ |
|---|---|---|---|---|
| $\begin{cases} \mathrm{pdf}_0 & \text{if } a_0 \le \lambda \le a_1 \\ \vdots & \vdots \\ \mathrm{pdf}_{m-1} & \text{if } a_{m-1} \le \lambda \le a_m \\ 0 & \text{otherwise} \end{cases}$ | Poisson $(\lambda T)$ | $\begin{cases} \frac{\mathrm{pdf}_{m-1}\cdot\frac{(\lambda T)^k}{k!}\cdot e^{-\lambda T}}{\sum_{i=0}^{m-1}\frac{\mathrm{pdf}_i}{T}(\gamma_1(a_{i+1}T)-\gamma_1(a_iT))} & \text{if } a_{m-1} \le \lambda \le a_m \\ 0 & \text{otherwise} \end{cases}$ | $\frac{(k+1)}{T}\cdot\frac{\sum_{i=0}^{m-1}\mathrm{pdf}_i\cdot(\gamma_2(a_{i+1}T)-\gamma_2(a_iT))}{\sum_{i=0}^{m-1}\mathrm{pdf}_i(\gamma_1(a_{i+1}T)-\gamma_1(a_iT))}$ | $\frac{(k+2)(k+1)}{T^2}\cdot\frac{\sum_{i=0}^{m-1}\mathrm{pdf}_i\cdot(\gamma_3(a_{i+1}T)-\gamma_3(a_iT))}{\sum_{i=0}^{m-1}\mathrm{pdf}_i(\gamma_1(a_{i+1}T)-\gamma_1(a_iT))}-(E[\lambda \mid k])^2$ |

Where $k$ is the number of failures, $n$ is the number of trials (demands). $I_1$, $I_2$, and $I_3$ are the respective cdfs of a $Beta(\alpha_1,\beta_1)$, $Beta(\alpha_2,\beta_1)$, and $Beta(\alpha_3,\beta_1)$ density functions. $\gamma_1$, $\gamma_2$, and $\gamma_3$ are the respective cdfs of a $Gamma(\alpha_1,\beta_2)$, $Gamma(\alpha_2,\beta_2)$, and $Gamma(\alpha_3,\beta_2)$ density functions, and $\alpha_1=k+1$, $\alpha_2=k+2$, $\alpha_3=k+3$, $\beta_1=n-k+1$, and $\beta_2=1$.

# 4 CONCLUSION

Bayesian update technique is highly-utilized in reliability analysis in general, and in PSA applications in specific. A fast and robust Bayesian update technique was developed and implemented at the Leibstadt nuclear power plant KKL. The developed tool succeeded to solve very ill-conditioned problems, where other tools usually fail.

Furthermore, based on the international generic data distribution type (prior distribution) and plant-specific evidence (likelihood distribution), the tool performs an exact analytical Bayesian update where possible. In cases where an analytical solution does not exists, it resorts to a very robust numerical technique, the Modal Method, which predicts the behavior of the posterior distribution, smartly discretizes, and integrates accordingly.

The tool proved to have better numerical integration abilities than the MATLAB built-in integral function, which experiences some numerical instabilities when calculating the posterior parameters (i.e. mean value, variance and percentiles).

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